

Injectivity of the specialization homomorphism of elliptic curves

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Abstract

Let $E : y^2 = x^3 + Ax^2 + Bx + C$ be a nonconstant elliptic curve over $\mathbb{Q}(t)$ with at least one nontrivial $\mathbb{Q}(t)$ -rational 2-torsion point. We describe a method for finding $t_0 \in \mathbb{Q}$ for which the corresponding specialization homomorphism $t \mapsto t_0 \in \mathbb{Q}$ is injective. The method can be directly extended to elliptic curves over $K(t)$ for a number field K of class number 1, and in principal for arbitrary number field K . One can use this method to calculate the rank of elliptic curves over $\mathbb{Q}(t)$ of the form as above, and to prove that given points are free generators. In this paper we illustrate it on some elliptic curves over $\mathbb{Q}(t)$ from an article by Mestre.

1 Introduction

Let

$$E = E(t) : y^2 = x^3 + Ax^2 + Bx + C \quad (1.1)$$

be a nonconstant (non-isotrivial) elliptic curve over $\mathbb{Q}(t)$, i.e., E is not isomorphic over $\mathbb{Q}(t)$ to an elliptic curve over \mathbb{Q} . For the sake of simplicity we will assume that $A, B, C \in \mathbb{Z}[t]$. It is known that the set $E(\mathbb{Q}(t))$ of $\mathbb{Q}(t)$ -rational points of E is finitely generated. Let D denote the discriminant of the polynomial $f(x) := x^3 + Ax^2 + Bx + C$. We note that $D \in \mathbb{Z}[t]$. Let $t_0 \in \mathbb{Q}$ be such that $D(t_0) \neq 0$. Then by specializing t to t_0 the specialization $E(t_0)$ of $E(t)$ is an elliptic curve over \mathbb{Q} and we have a specialization homomorphism $\sigma = \sigma_{t_0} : E(\mathbb{Q}(t)) \rightarrow E(t_0)(\mathbb{Q})$ (note that it is well defined). For more on this topic see [Sil4, Appendix C §20]. The specialization homomorphism can be defined for general non-split elliptic surfaces and in a more general situation. In 1952 A. Néron [Né] showed that the specialization fails to be injective for $t_0 \in \mathbb{Q}$ on a small subset (of density 0) (see [Se, Section 11.1]). J.

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H. Silverman [Sil1, Sil2] in 1983 using heights and J. Top in 1985 in his master's thesis (see [To]) by extending Néron's techniques proved the so called Silverman specialization theorem, which says that the specialization homomorphism is in fact injective for all but finitely many rational t_0 . As far as we know, there is no practical algorithm for determining such a t_0 (for general non-split elliptic surfaces). As we learned from J. H. Silverman, all constants in [Sil2], Section 4, Theorem B, and Section 5, Theorem C can, in principal, be effectively computed. Therefore, one can find a computable constant C , such that for all algebraic t_0 with height greater than C , the specialization homomorphism at t_0 is injective. However, the constants are too large to be practical. Similarly for methods from [Sil3]. In this paper we use the ideas from Néron and Top (which also appear in [Ha]). We obtain a method for finding a specialization $t \mapsto t_0 \in \mathbb{Q}$ such that the specialization homomorphism is injective, in the case of elliptic curves of shape (1.1) having at least one non-trivial $\mathbb{Q}(t)$ -rational 2-torsion point. This improves and extends the method from [GT1]. Let us state the main results (see Section 2 and Section 3 for the proofs):

Theorem 1.1 *Let E be a nonconstant elliptic curve over $\mathbb{Q}(t)$, given by the equation*

$$E = E(t) : y^2 = (x - e_1)(x - e_2)(x - e_3), (e_1, e_2, e_3 \in \mathbb{Z}[t]).$$

Assume that $t_0 \in \mathbb{Q}$ satisfies the following condition.

(A) For every nonconstant square-free divisor h in $\mathbb{Z}[t]$ of

$$(e_1 - e_2) \cdot (e_1 - e_3) \quad \text{or} \quad (e_2 - e_1) \cdot (e_2 - e_3) \quad \text{or} \quad (e_3 - e_1) \cdot (e_3 - e_2),$$

the rational number $h(t_0)$ is not a square in \mathbb{Q} .

Then the specialization homomorphism $\sigma : E(\mathbb{Q}(t)) \rightarrow E(t_0)(\mathbb{Q})$ is injective.

This leads to a practical criterion that can be directly extended to number fields K of class number one, where the elliptic curves are as in Theorem 1.1 with $e_j \in \mathcal{O}_K[t]$ (here \mathcal{O}_K is the ring of integers of K). The criterion can be extended to arbitrary number fields. However, the calculations over general number fields are rather complicated. For example, if the class number of the field K is greater than 1, the ring of integers \mathcal{O}_K has to be replaced by a suitable UFD.

Theorem 1.2 *Let K be a number field. Let \mathcal{R}_K be a chosen unique factorization domain such that $\mathcal{O}_K \subset \mathcal{R}_K \subset K$ (and such that its group of units is finitely generated). If K is of class number one we always choose $\mathcal{R}_K = \mathcal{O}_K$. Let E be a nonconstant elliptic curve over $K(t)$, given by the equation*

$$E = E(t) : y^2 = (x - e_1)(x - e_2)(x - e_3), (e_1, e_2, e_3 \in \mathcal{R}_K[t]). \quad (1.2)$$

Assume that $t_0 \in K$ satisfies the following condition.

(C) For every nonconstant square-free divisor h in $\mathcal{R}_K[t]$ of

$$(e_1 - e_2) \cdot (e_1 - e_3) \quad \text{or} \quad (e_2 - e_1) \cdot (e_2 - e_3) \quad \text{or} \quad (e_3 - e_1) \cdot (e_3 - e_2),$$

the algebraic number $h(t_0)$ is not a square in K .

Then the specialization homomorphism $\sigma : E(K(t)) \rightarrow E(t_0)(K)$ is injective.

The first author extends Theorem 1.1 to elliptic curves of shape (1.1) having exactly one non-trivial $\mathbb{Q}(t)$ -rational 2-torsion point. Let us set $f(x) := x^3 + Ax^2 + Bx + C = (x - e_1)(x - e)(x - \bar{e})$ where $e_1 \in \mathbb{Z}[t]$ and e, \bar{e} are conjugate. Here $D = (e_1^2 - (e + \bar{e})e_1 + e\bar{e})^2(e - \bar{e})^2$.

Theorem 1.3 *Let $E = E(t) : y^2 = x^3 + Ax^2 + Bx + C$; $A, B, C \in \mathbb{Z}[t]$ be a nonconstant elliptic curve over $\mathbb{Q}(t)$. Assume that E has exactly one nontrivial 2-torsion point over $\mathbb{Q}(t)$, i.e. that $x^3 + Ax^2 + Bx + C = (x - e_1)(x - e)(x - \bar{e})$ where $e_1 \in \mathbb{Z}[t]$ and e, \bar{e} are conjugate. Let $t_0 \in \mathbb{Q}$ satisfy the following condition:*

(A) For every nonconstant square-free divisor h of $(e_1^2 - (e + \bar{e})e_1 + e\bar{e})$ or $(e - \bar{e})^2$ in $\mathbb{Z}[t]$ the rational number $h(t_0)$ is not a square in \mathbb{Q} .

Then the specialized curve E_{t_0} is elliptic and the specialization homomorphism at t_0 is injective.

This theorem also can be directly extended to number fields K of class number one. It can also be extended to arbitrary number fields by passing from the ring of integers \mathcal{O}_K to a suitable ring \mathcal{R}_K as above.

Theorem 1.4 *Let K be a number field. Let \mathcal{R}_K be a chosen unique factorization domain such that $\mathcal{O}_K \subset \mathcal{R}_K \subset K$ (and such that its group of units is finitely generated). If K is of class number one we always choose $\mathcal{R}_K = \mathcal{O}_K$. Let E be a nonconstant elliptic curve over $K(t)$, given by the equation $E : y^2 = x^3 + Ax^2 + Bx + C$; $A, B, C \in \mathcal{R}_K[t]$. Assume that E has exactly one nontrivial 2-torsion point over $K(t)$, i.e. that $x^3 + Ax^2 + Bx + C = (x - e_1)(x - e)(x - \bar{e})$ where $e_1 \in \mathcal{R}_K[t]$ and e, \bar{e} are conjugate. Let $t_0 \in K$ satisfy the following condition:*

(A) For every nonconstant square-free divisor h of $(e_1^2 - (e + \bar{e})e_1 + e\bar{e})$ or $(e - \bar{e})^2$ in $\mathcal{R}_K[t]$ the number $h(t_0)$ is not a square in K .

Then the specialized curve E_{t_0} is elliptic and the specialization homomorphism at t_0 is injective.

The criterion from these theorems has no direct extension to general elliptic curves (see Remark 3.4).

There are a few methods for calculating the rank of specific types of elliptic curves over $\mathbb{Q}(t)$ such as the method based on the Tate-Shioda formula (see [Sh1], Corollary 15, and [Sh2]), Nagao's conjectural method [Na] and the 2-descent method (see, for example [Br]). Our method helps in calculating the rank of elliptic curves over $\mathbb{Q}(t)$ by using information about a suitably chosen specialized curve. One can use it even to prove that a set of points are free generators. In Section 4 we describe and comment on a family of quadratic twists coming from Mestre: a family of quadratic twists of the general family of elliptic curves $E = E^{a,b} : y^2 = x^3 + ax + b$

over \mathbb{Q} with certain 14th degree polynomials $g = g^{a,b}$ in a variable t over \mathbb{Q} . It is known that the rank of E_g over $\mathbb{Q}(t)$ is at least 2 for all a, b , $ab \neq 0$. By a general principle, these ranks are at most 6. We performed an extensive calculation using our criterion for number fields of class number one (including \mathbb{Q}) and for a number field of class number two. In all cases we get that the rank is two and prove that the given points are free generators. Two examples are presented in details, one when the splitting field of the polynomial $x^3 + ax + b$ is a sextic field with class number one (see Example 4.3), the other when the splitting field has class number two (see Example 4.4). We used Magma [MG], Pari [Par], and `mwrank` [MW] for most of our computations.

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2 Elliptic curve with rational 2-torsion

In this section we prove Theorem 1.1 and sketch a proof of Theorem 1.2. First we work over \mathbb{Q} as the field of constants. At the end of the section we extend the consideration to arbitrary number fields. Let E be a nonconstant elliptic curve over $\mathbb{Q}(t)$, given by the equation

$$E = E(t) : y^2 = (x - e_1)(x - e_2)(x - e_3), (e_1, e_2, e_3 \in \mathbb{Z}[t]).$$

We have homomorphisms $\Theta_i : E(\mathbb{Q}(t)) \rightarrow \mathbb{Q}(t)^\times / (\mathbb{Q}(t)^\times)^2$, $i = 1, 2, 3$ given by

$$\begin{cases} \Theta_i(x, y) = (x - e_i) \cdot (\mathbb{Q}(t)^\times)^2, & \text{if } x \neq e_i, \\ \Theta_i(e_i, 0) = (e_j - e_i)(e_k - e_i) \cdot (\mathbb{Q}(t)^\times)^2, & \text{where } i \neq j \neq k \neq i, \\ \Theta_i(O) = 1 \cdot (\mathbb{Q}(t)^\times)^2, & \text{(here } O \text{ denotes the neutral element).} \end{cases}$$

Lemma 2.1 $P \in 2E(\mathbb{Q}(t))$ if and only if $\Theta_i(P) = 1 \cdot (\mathbb{Q}(t)^\times)^2$ for $i = 1, 2, 3$.

Proof. The claim follows from [Hu], Chapter 1, Theorem (4.1), and Chapter 6, Proposition (4.3). See also [Ha, Section 1] for the generalization to hyperelliptic curves. ■

Since $\mathbb{Z}[t]$ is a unique factorization domain (UFD), it is evident that for each $P \in E(\mathbb{Q}(t))$ there exists exactly one triple (s_1, s_2, s_3) , $s_i = s_i(P) \in \mathbb{Z}[t]$, $i = 1, 2, 3$, of non-zero square-free elements from $\mathbb{Z}[t]$, such that

$$\Theta_i(P) = s_i(P) \cdot (\mathbb{Q}(t)^\times)^2. \tag{2.1}$$

We will also use notation $s_i(t)$ for s_i . Lemma 2.1 can be reformulated as

$$P \in 2E(\mathbb{Q}(t)) \text{ if and only if } s_i(P) = 1, \text{ for } i = 1, 2, 3. \quad (2.2)$$

It is easy to see that $s_1 s_2 s_3 \in \mathbb{Z}[t]^2$, and that, for each i and each prime $p \in \mathbb{Z}[t]$, we have

$$\text{if } p|s_i \text{ then } p|s_j s_k, \text{ where } i \neq j \neq k \neq i. \quad (2.3)$$

Let $P \in E(\mathbb{Q}(t)) \setminus \{O\}$. Then the first coordinate of P is of the form

$$x(P) = \frac{p(t)}{q(t)^2}, \text{ with } p(t), q(t) \in \mathbb{Z}[t] \text{ coprime} \quad (2.4)$$

(recall that $\mathbb{Z}[t]$ is an UFD). Therefore $p(t) - e_i(t)q^2(t) = s_i(P)\square_{\mathbb{Z}[t]}$, $i = 1, 2, 3$, where $\square_{\mathbb{Z}[t]}$ denotes a square of an element of $\mathbb{Z}[t]$. By this, (2.3) and the fact that s_i are square-free, we deduce that

$$s_i|(e_j - e_i)(e_k - e_i), \text{ where } i \neq j \neq k \neq i \quad (2.5)$$

for each i (see also [Hu], Chapter 6, Proposition (4.1)). For example, a prime factor of s_1 is also a prime factor of $s_2 s_3$. Assume that it is a prime factor of s_2 . Then it is a prime factor of $(e_1 - e_2)q^2(t)$, hence it is a prime factor of $e_1 - e_2$.

In Theorem 1.1 we make a refinement of the method from [GT1], Theorem 3.2. The proof is a modification of that proof. Now we present the proof of Theorem 1.1.

Proof of Theorem 1.1. Note that the specialized curve is non-singular (see Lemma 3.1 (i)). Let us prove that the specialization homomorphism is injective. Assume that the conditions of the theorem are satisfied and that σ is not an injection. So there exists a point $P \in E(\mathbb{Q}(t)) \setminus \{O\}$ such that $\sigma(P) = O$. We will prove that it leads to a contradiction. First we prove that $P \in 2E(\mathbb{Q}(t))$. By (2.2), it is equivalent to proving that $s_i(t) = 1$ for each $i = 1, 2, 3$. Since σ is injective on the torsion part [Sil5, p. 272–273, proof of Theorem III.11.4], we may assume that $P \neq (e_i, 0)$, $i = 1, 2, 3$. By $p(t) - e_k(t)q^2(t) = s_k(P)\square_{\mathbb{Z}[t]}$ and the fact that $q(t_0) = 0$, we get $p(t_0) = s_k(t_0)\square_{\mathbb{Q}}$. Since $p(t_0)$ should be a non-zero rational square (recall that $q(t_0) = 0$ and p, q are coprime), we see that $s_i(t_0)$ is a rational square, for each $i = 1, 2, 3$. We claim that $s_k(t) = 1$ for each $k = 1, 2, 3$, i.e., that $P \in 2E(\mathbb{Q}(t))$.

Assume that $s_k(t)$ is non-constant for some k . By the above discussion $s_k(t_0)$ is a rational square, which is in contradiction with condition (A) of the theorem (recall that by (2.5), s_k is a nonconstant square-free divisor of $(e_i - e_k) \cdot (e_j - e_k)$ in $\mathbb{Z}[t]$, with $i \neq j \neq k \neq i$). Therefore $s_k(t)$ is constant for each k . Since $s_k(t)$ is square-free in $\mathbb{Z}[t]$ and $s_k(t_0)$ is a rational square, we see that $s_k(t) = 1$, for each k . This proves that $P \in 2E(\mathbb{Q}(t))$.

We claim that there is $P_1 \in E(\mathbb{Q}(t))$ such that $2P_1 = P$ and $\sigma(P_1) = O$. Let $P'_1 \in E(\mathbb{Q}(t))$ be any point with $2P'_1 = P$. Then $2\sigma(P'_1) = O$, i.e., $\sigma(P'_1)$ is a 2-torsion point on the specialized curve. Since σ is injective on the torsion points, there exists a 2-torsion point $Q \in E(\mathbb{Q}(t))$ such that $\sigma(Q) = \sigma(P'_1)$. Put $P_1 = P'_1 - Q$. Then $2P_1 = P$, especially $P_1 \neq O$, and $\sigma(P_1) = O$. Note that P_1 is of infinite

order. Now the procedure can be continued with P_1 instead of P , the contradiction. Therefore $P = O$, i.e., σ is injective. ■

Remark 2.2 *Condition (A) in Theorem 1.1 is weaker than the following condition (A') For every nonconstant square-free divisor h in $\mathbb{Z}[t]$ of*

$$(e_1 - e_2) \cdot (e_2 - e_3) \cdot (e_3 - e_1)$$

the rational number $h(t_0)$ is not a square in \mathbb{Q} .

For example, set $e_1 = 0$, $e_2 = t$, $e_3 = 7t + 1$. Then $t_0 := \frac{1}{21}$ satisfies condition (A). Since $\frac{1}{21}(6 \cdot \frac{1}{21} + 1)(7 \cdot \frac{1}{21} + 1) = (\frac{2}{7})^2$, it does not satisfy condition (A').

Remark 2.3 *Let \mathcal{T} denote the set of all integers t_0 that satisfy Condition (A) from Theorem 1.1. Then there is an effectively computable constant $c > 0$ such that $\mathcal{T} \cap [-c, c] \neq \emptyset$. Namely, condition (A) in Theorem 1.1 produces the equations of the form $z^2 = h(t)$ for certain square-free polynomials h over \mathbb{Z} of degree $d \geq 1$. If $d \leq 2$, the corresponding curve has genus 0, if $d = 3$ or 4 the genus is one, and if $d \geq 5$ the curve is hyperelliptic with genus ≥ 2 . Recall that curves over \mathbb{Q} of genus at least 1 have only finitely many integer points. Moreover, for elliptic and hyperelliptic curves, there are explicit bounds for the height of integer points ([Ba], [Bu], Theorem 1; see also [ES], Theorem 1 b, for a bound of the number of integer points). If $d = 1$ or $d = 2$ then the curve $z^2 = h(t)$ may have finitely many or infinitely many integer points. The case $d = 1$ is straightforward, while the case $d = 2$ reduces to an estimating of the number of integer solutions for $Dz^2 = t^2 + B$ where D is a square-free integer and B a nonzero integer. The most demanding case is when $D \geq 2$. Then there is an effectively computable constant $c_1 = c_1(D, B)$ such that $Dz^2 = t^2 + B$ has $\leq c_1 \tau(B) \log X$ integer solutions with $|t|, |z| \leq X$ for sufficiently large X , where $\tau(B)$ denotes the number of positive divisors of B (see [PZ], Lemma 3. for a more precise estimation). ■*

Now we sketch a proof of Theorem 1.2. Assume that K is an arbitrary number field with the ring of integers \mathcal{O}_K . There exist at least one unique factorization domain \mathcal{R}_K , $\mathcal{O}_K \subset \mathcal{R}_K \subset K$ such that its group of units is finitely generated (see for example [Kn, p. 94, p. 127]).

Proof of Theorem 1.2. Relations (2.1)–(2.5) remain valid after replacing $\mathbb{Z}[t]$ by $\mathcal{R}_K[t]$. Now the proof is analogous to the proof of Theorem 1.1. Note that the theorem remains valid even if we exclude the condition that \mathcal{R}_K^\times is finitely generated. However, this condition reduces the checking of Condition (C) from the Theorem to checking of only a finitely many square-free divisors h in $\mathcal{R}_K[t]$. ■

It can be seen that there is a variant of Remark 2.3 for elliptic curves of the form (1.2). In the following remark we use another argument to prove that there are a lot of rational integers t_0 satisfying condition (C) from Theorem 1.2.

Remark 2.4 According to [Sch], Section 5, Definition 24, Theorem 50 and Corollary 1, for each $F \in \mathbb{C}[z, t]$ either:

- (i) every congruence class \mathcal{C} in \mathbb{Z} contains a congruence subclass \mathcal{C}^* such that for all $t_0 \in \mathcal{C}^*$ the polynomial $F(z, t_0)$ has no zero in K , or
- (ii) F viewed as a polynomial in z has a zero in $K(t)$.

By consecutively applying this to the polynomials $F[z, t] := z^2 - h(t)$ above, we see that for each congruence class \mathcal{C} in \mathbb{Z} there exists a congruence subclass \mathcal{C}^* of \mathcal{C} , such that the conditions from Theorem 1.2 are satisfied for all $t_0 \in \mathcal{C}^*$.

3 Elliptic curves with exactly one rational 2-torsion point

In this section we prove Theorem 1.3, which extends the criterion for injectivity from Theorem 1.1 to elliptic curves having exactly one nontrivial rational 2-torsion point. After that we sketch a proof of Theorem 1.4. Similarly as in Section 2 we first work over \mathbb{Q} as the field of constants and after that we extend the consideration to arbitrary number fields. Recall that $E : y^2 = x^3 + Ax^2 + Bx + C$, $A, B, C \in \mathbb{Z}[t]$ is a non-constant elliptic curve over $\mathbb{Q}(t)$, and that D denotes the discriminant of the polynomial $f(x) := x^3 + Ax^2 + Bx + C$. Since we assume that E has exactly one $\mathbb{Q}(t)$ -rational point we may work with the equation

$$E : y^2 = x^3 + Ax^2 + Bx, \quad A, B \in \mathbb{Z}[t], \quad A^2 - 4B \notin \mathbb{Z}[t]^2, \quad (3.1)$$

with $D = B^2(A^2 - 4B)$. Note that the factor B corresponds to $e_1^2 - (e + \bar{e})e_1 + e\bar{e}$ from Theorem 1.3, while the factor $A^2 - 4B$ corresponds to $(e - \bar{e})^2$. The following condition concerning the discriminant D and a rational number t_0 is a reformulation of the condition from Theorem 1.3.

(A) For each factor h of B or of $A^2 - 4B$ in $\mathbb{Z}[t]$ if $h(t_0)$ is a square in \mathbb{Q} , then h is a square in $\mathbb{Z}[t]$.

Lemma 3.1 Assume that a rational number t_0 satisfies condition (A). Then:

- (i) $D(t_0) \neq 0$.
- (ii) The polynomial $f(x, t_0) := x^3 + A(t_0)x^2 + B(t_0)x$ has exactly one \mathbb{Q} -rational root.

Proof. (i) Contrary, D has a linear factor h with $h(t_0) = 0$, which contradicts condition (A).

(ii) Since $A^2 - 4B$ is not a square in $\mathbb{Z}[t]$, we conclude by (A) that $A(t_0)^2 - 4B(t_0)$ is not a square in \mathbb{Q} . ■

For the proof of Theorem 1.3 we will need some wellknown facts on 2-isogeny (see, for example, [Hu], Chapter 4, Section 5). Let us define

$$\bar{E} : y^2 = x^3 - 2Ax^2 + (A^2 - 4B)x, \quad (3.2)$$

and let \bar{D} denote the discriminant of the polynomial $x^3 - 2Ax^2 + (A^2 - 4B)x$. We have $\bar{D} = 16B(A^2 - 4B)^2$. The map $\phi : E \rightarrow \bar{E}$ defined on points different from O and $(0,0)$ by

$$\phi(P) = \left(\frac{y(P)^2}{x(P)^2}, \frac{y(P)(x(P)^2 - B)}{x(P)^2} \right),$$

gives rise to an isogeny of degree two with kernel $\{O, (0,0)\}$ and with the dual isogeny defined by

$$\psi(\bar{P}) = \left(\frac{y(\bar{P})^2}{4x(\bar{P})^2}, \frac{y(\bar{P})(x(\bar{P})^2 - (A^2 - 4B))}{8x(\bar{P})^2} \right).$$

Further, $\bar{P} \in \bar{E}(\mathbb{Q}(t))$ different from O and $(0,0)$ is in $\phi(E(\mathbb{Q}(t)))$ if and only if $x(\bar{P})$ is a square from $\mathbb{Q}(t)$ (see [Hu], Chapter 4, Proposition 5.7 or [SiTa], p.83-85). From this it is easy to see that $P \in E(\mathbb{Q}(t))$ different from O and $(0,0)$ is in $\psi(\bar{E}(\mathbb{Q}(t)))$ if and only if $x(P)$ is a square from $\mathbb{Q}(t)$.

Proof of Theorem 1.3. By Lemma 3.1 (i), the specialized curve is non-singular, hence it is an elliptic curve over \mathbb{Q} . It remains to prove that the specialization homomorphism is injective. We may assume that E is given by the equation of shape (3.1). Assume that $P \in E(\mathbb{Q}(t))$ is nontrivial with trivial specialization at t_0 . We will see that this leads to the contradiction. Let us write $P = (x(P), y(P))$, where

$$x(P) = \frac{p}{q^2} \tag{3.3}$$

with coprime p, q from $\mathbb{Z}[t]$, especially $q(t_0) = 0$ and $p(t_0)$ is a square in \mathbb{Q} .

We claim that $P \in 2E(\mathbb{Q}(t))$. To prove this claim we first prove that $x(P)$ is a square in $\mathbb{Q}(t)$. It is enough to prove that p is a square in $\mathbb{Z}[t]$. Contrary, p has a prime π in $\mathbb{Z}[t]$ with an odd multiplicity. Since $p(p^2 + Apq^2 + Bq^4)$ is a square in $\mathbb{Z}[t]$, we see that π is also a prime factor of B . Therefore, the square-free part h of p in $\mathbb{Z}[t]$ has at least one prime factor and h divides B . By condition (A) of the Theorem $h(t_0)$ is not square in \mathbb{Q} . On the other side, since $p(t_0)$ is a square in \mathbb{Q} and $q(t_0) = 0$ we get that $h(t_0)$ is a square in \mathbb{Q} . This is the contradiction, hence $x(P)$ is a square in $\mathbb{Q}(t)$ (note that $-p$ is not a square in $\mathbb{Z}[t]$). From this we see that there exists a point $\bar{P} \in \bar{E}(\mathbb{Q}(t))$ such that $\psi(\bar{P}) = P$ (see the text after the proof of Lemma 3.1). It is easy to see that the specialization of \bar{E} at t_0 is an elliptic curve and that \bar{P} specializes to O or to $(0,0)$. If \bar{P} specializes to $(0,0)$ then $\bar{P} + (0,0)$ specializes to O , so we may assume that \bar{P} specializes to O . Using the argument from the first part of the proof we conclude that $x(\bar{P})$ is a square in $\mathbb{Q}(t)$, and further that there is a point $P_1 \in E(\mathbb{Q}(t))$ such that $\bar{P} = \phi(P_1)$. Finally, we get $P = \psi(\bar{P}) = \psi(\phi(P_1)) = 2P_1$, as we claimed.

The rest of the proof is analogous to the end of the proof of Theorem 1.1. We only have to note that, by Lemma 3.1 (ii), the specialized curve E_{t_0} has exactly one non-trivial \mathbb{Q} -rational 2-torsion point. ■

We support the criterion from Theorem 1.3 by two examples.

Example 3.2 Let E be the elliptic curve over $\mathbb{Q}(t)$ given by the equation $y^2 = x^3 + t^2x^2 - x$. Using the Tate-Shioda formula (see [Sh2], Corollary 5.3 and Lemma 10.1), one can find that it has rank 1 over $\mathbb{Q}(t)$ with the point $P = (1, t)$ of infinite order. By computing mP for $m = 2, 3, \dots, 12$ we see that the specialization is injective for all rational t_0 except $t = 0, \pm 1$. This is not in collision with Theorem 1.3 because $t = 0, \pm 1$ do not satisfy condition (A).

Example 3.3 Let E be an elliptic curve X over $\mathbb{Q}(t)$ given by the equation $y^2 = x(x^2 - 2(5(2t^2 - 2t + 1)(t^2 - 2t + 2) - 2(t^2 - 1)^2)x + 25(2t^2 - 2t + 1)^2(t^2 - 2t + 2)^2)$ (see [Br], p. 551, formula (14')). By a detailed 2-descent analysis A. Bremner proved that the rank of E over $\mathbb{Q}(i)(t)$ is three, especially the rank of E over $\mathbb{Q}(t)$ is at most three. He presented three independent $\mathbb{Q}(t)$ -rational points, which implies that the rank of E over $\mathbb{Q}(t)$ is exactly three. To illustrate the criterion from Theorem 1.3 let us note that the discriminant of E equals to

$$-2^8 \cdot 5^4 \cdot (t - 1)^2(t + 1)^2(9t^4 - 30t^3 + 47t^2 - 30t + 9)(t^2 - 2t + 2)^4(2t^2 - 2t + 1)^4.$$

Further, E has a nontrivial 2-torsion point $(0, 0)$ and two 2-torsion points conjugate over $\mathbb{Q}(t, \sqrt{-9t^4 + 30t^3 - 47t^2 + 30t - 9})$. Since $t_0 = \frac{5}{2}$ satisfies condition (A), and the specialized curve has rank three over \mathbb{Q} , we conclude, by Theorem 1.3, that the rank of E over $\mathbb{Q}(t)$ is at most three.

In the following remark we present two examples which show that the criterion from Theorem 1.3 has not a direct extension to general elliptic curves over $\mathbb{Q}(t)$ i.e. elliptic curves $E : y^2 = x^3 + Ax^2 + Bx + C$, where $f(x) := x^3 + Ax^2 + Bx + C$ is irreducible over $\mathbb{Q}(t)$. Let us consider the condition

(\mathcal{A}_∞) For each factor h of D in $\mathbb{Z}[t]$ if $h(t_0)$ is a square in \mathbb{Q} , then h is a square in $\mathbb{Z}[t]$.

It is clear that for polynomials f having at least one nontrivial rational root, if t_0 satisfies condition (A) or condition (\mathcal{A}), then it satisfies condition (\mathcal{A}_∞). For polynomials f that are irreducible over $\mathbb{Q}(t)$ we consider the additional condition: (\mathcal{B}) The polynomial $f(x, t_0) := x^3 + A(t_0)x^2 + B(t_0)x + C(t_0)$ is irreducible over \mathbb{Q} .

Remark 3.4 • Let E be the elliptic curve over $\mathbb{Q}(t)$ given by the equation $y^2 = x^3 - x + t^2$. It has rank 2 over $\mathbb{Q}(t)$ with trivial torsion and generators any two of the points $(0, t), (1, t), (-1, t)$ (see [Sh3], Theorem (A_2)). Further its discriminant equals to $16(4 - 27t^4)$ and the Galois group of the polynomial $x^3 - x + t^2$ is isomorphic to the symmetric group S_3 . It is easy to see that $t_0 = \pm 1, \pm \frac{1}{2}$ satisfy conditions (\mathcal{A}_∞) and (\mathcal{B}). On the other side, one can check that the specializations at $\pm 1, \pm \frac{1}{2}$ are not injective.

- Let E be the elliptic curve over $\mathbb{Q}(t)$ given by the equation $y^2 = x^3 - t^2x + 1$. It has rank 3 over $\mathbb{Q}(t)$ with trivial torsion and generators $(0, 1), (-1, t), (-t, 1)$ (see [Ta], Proposition 2.1 (iv)). Its discriminant equals to $16(4t^6 - 27)$ and the

Galois group of the polynomial $x^3 - t^2x + 1$ is isomorphic to the symmetric group S_3 . It is easy to see that $t_0 = 0$ satisfies condition (\mathcal{A}_∞) , while $t_0 = \pm 1, \pm 2$ satisfy conditions (\mathcal{A}_∞) and (\mathcal{B}) . On the other side, one can check that the specializations at $0, \pm 1 \pm 2$ are not injective.

Now we sketch a proof of Theorem 1.4. First note that the facts on 2-isogeny that we used in the proof of Theorem 1.3 are valid for elliptic curves over arbitrary fields of characteristic zero. Also condition (\mathcal{A}) can be extended directly to elliptic curves over $K(t)$ for a number field K of class number 1 (we only have to replace $\mathbb{Z}[t]$ by $\mathcal{O}_K[t]$). Now the proof of Theorem 1.3 can be extended directly to this case (all we need is the unique factorization in $\mathcal{O}_K[t]$). It is clear that the later proof can be extended to arbitrary number fields K (we only have to replace $\mathcal{O}_K[t]$ by $R_K[t]$).

4 Application to an example by Mestre

In this section we apply our criterion to a family of quadratic twists of Mestre from the following example.

Example 4.1 ([Me], [RS, Theorem 3.7], [ST, Theorem 3]) *Let $a, b \in \mathbb{Q}$ such that $ab \neq 0$, let*

$$g(t) = g^{a,b}(t) = -ab \cdot (t^2 + 1) \cdot (b^2(t^4 + t^2 + 1)^3 + a^3t^4(t^2 + 1)^2)$$

and let $E = E^{a,b}$ be the elliptic curve over \mathbb{Q} given by the equation

$$y^2 = x^3 + ax + b. \tag{4.1}$$

Then $E_g = E_g^{a,b} : y^2 = x^3 + ag(t)^2x + bg(t)^3$ has rank at least 2 over $\mathbb{Q}(t)$, with two independent points $P = P_g^{a,b}$ and $Q = Q_g^{a,b}$ with coordinates

$$P = \left(-\frac{b(t^2 + t + 1)(t^2 - t + 1)}{a(t^2 + 1)} \cdot g(t), \frac{1}{a^2(t^2 + 1)^2} \cdot g(t)^2 \right) \tag{4.2}$$

$$Q = \left(-\frac{b(t^2 + t + 1)(t^2 - t + 1)}{a t^2(t^2 + 1)} \cdot g(t), \frac{1}{a^2 t^3(t^2 + 1)^2} \cdot g(t)^2 \right). \tag{4.3}$$

Similarly as in [ST], Section 4, let C be a smooth complete model over \mathbb{Q} of the curve given by $s^2 = g(t)$. Then for each point $T = (x(t), y(t)) \in E_g(\mathbb{Q}(t))$ there is a morphism $\phi_T : C \rightarrow E$ defined by $\phi_T(t, s) = (\frac{x(t)}{s^2}, \frac{y(t)}{s^3})$. This gives rise to an isomorphism between $\text{Mor}_{\mathbb{Q}}(C, E)$ modulo constant morphisms and $E_g(\mathbb{Q}(t))$ modulo torsion. The degree $\deg \phi_T$ equals to $\deg \frac{x(t)}{g(t)}$. Since $\deg(2\phi) = 4 \deg \phi$ for each $\phi \in \text{Mor}(C, E)$ the mapping $T \mapsto \frac{1}{2} \deg \phi_T$ is the canonical height on $E_g(\mathbb{Q}(t))$.

(for a direct proof see [GL]). Let $\langle \cdot, \cdot \rangle$ denote the corresponding canonical bilinear form on $E_g(\mathbb{Q}(t)) \times E_g(\mathbb{Q}(t))$, i.e.

$$\langle T, S \rangle = \frac{1}{2}(\deg \phi_{T+S} - \deg \phi_T - \deg \phi_S), \text{ for each } T, S \in E_g(\mathbb{Q}(t)),$$

especially $\deg \phi_T = \langle T, T \rangle$, for each T . This approach works for general nonconstant twists (quadratic or else) of constant elliptic curves. See [Pa, §4.2, §4.4] for some further aspects of this approach which is specific for the Mestre family.

Since $\deg \phi_P = \deg \phi_Q = 4$ and $\deg \phi_T \geq 4$ for all nontorsion points T , it is intuitively clear that P, Q generate a maximal nontorsion subgroup of rank two in $E_g(\mathbb{Q}(t))$. In the following lemma we give a precise formulation and a proof.

Lemma 4.2 *Let $E_g = E_g^{a,b}$, $P = P_g^{a,b}$, $Q = Q_g^{a,b}$ be as in Example 4.1. Then:*

- (i) *P, Q are free generators of each subgroup of rank two in $E_g(\mathbb{Q}(t))$ containing P, Q .*
- (ii) *Assume that there exists a $t_0 \in \mathbb{Q}$ for which the corresponding specialization homomorphism σ_{t_0} is injective and the rank of the specialized elliptic curve over \mathbb{Q} is two. Then the rank of $E_g(\mathbb{Q}(t))$ is two and P, Q are its free generators.*

Proof. (i) Let M be a torsion-free subgroup of $E_g(\mathbb{Q}(t))$ of rank two containing P, Q and let $T \in M$ be a nontorsion point. Then there is a nontrivial relation

$$kT = mP + nQ, \quad m, n, k \in \mathbb{Z}.$$

We may assume that $k > 0$. By consecutive adding or subtracting kP or kQ , it leads to

$$kT_1 = m'P + n'Q, \quad \text{with } -\frac{k}{2} \leq m', n' \leq \frac{k}{2}.$$

From this we get

$$k^2 \deg \phi_{T_1} = m'^2 \deg \phi_P + n'^2 \deg \phi_Q + 2m'n' \langle P, Q \rangle,$$

which provides an upper bound for $\deg \phi_{T_1}$:

$$\deg \phi_{T_1} \leq \frac{\deg \phi_P + \deg \phi_Q + 2|\langle P, Q \rangle|}{4}. \quad (4.4)$$

Since $\deg \phi_P = \deg \phi_Q = 4$, $\deg \phi_{P+Q} \leq 8$ and $\deg \phi_{P-Q} \leq 8$ for each a, b , we conclude, by the parallelogram law that $\deg \phi_{P+Q} = \deg \phi_{P-Q} = 8$, hence $\langle P, Q \rangle = 0$. By (4.4) we get $\deg \phi_{T_1} \leq 2$. We claim that $\deg \phi_{T_1} = 1$ or $\deg \phi_{T_1} = 2$ is impossible. Contrary, $x = x(T_1)/g(t) = \frac{\alpha(t)}{\beta(t)}$, where α, β are nonzero polynomials over \mathbb{Q} of degree at most 2 and at least one of them is non-constant. Plugging in $g(t)y^2 = x^3 + ax + b$, we get that there is a nonzero polynomial w over \mathbb{Q} of degree at most 6, such that $w(t)\beta(t)g(t)$ is a square in $\mathbb{Q}[t]$. It is impossible since $g(t)$ is squarefree with degree 14. Therefore $\deg \phi_{T_1} = 0$, hence T_1 is torsion, i.e. T is a

\mathbb{Z} -linear combination of P, Q and torsion points.

(ii) Directly from (i). ■

Let K denote the splitting field of the cubic polynomial $f(x) = x^3 + ax + b$, thus K is Galois. It is well known that either $K = \mathbb{Q}$, K is a quadratic field over \mathbb{Q} , K is a cubic field over \mathbb{Q} with cyclic Galois group, or K is a sextic field over \mathbb{Q} with the Galois group isomorphic to the symmetric group \mathbf{S}_3 .

In the sequel we illustrate Theorem 1.2 on two concrete examples from Mestre's family to get the rank and prove that given points are free generators, we describe the corresponding algorithm. In fact, we performed a more extensive calculation for various number fields K of class number one (including \mathbb{Q}) with various Galois groups, and for several elliptic curves for a number field K of class number two. In all cases we get that the rank is two and proved that the given points are free generators. One can find these calculations in [GT2, Section 5]. Two examples are presented here in details, one when the splitting field K is a sextic field with class number one (see Example 4.3), the other when K has class number two (see Example 4.4). Calculations are performed using a variety of packages: GP/Pari [Par], MAGMA [MG], mwrank [MW]. Let us sketch the algorithm. After fixing a concrete value of (a, b) we chose a rational number t_0 such that:

- σ_{t_0} is an injection (by using Theorem 1.2).
- t_0 is such that the specialized curve $E_g^{a,b}(t_0)$ is of rank 2 over \mathbb{Q} (which is calculated with `mwrank` and Magma's command `MordellWeilShaInformation`). To avoid extensive calculations, before calculating the rank we checked if the root number was one.

Now Lemma 4.2(ii) is applied to conclude that $E_g = E_g^{a,b}$ over $\mathbb{Q}(t)$ has actually free generators the two points P, Q from Example 4.1. If K properly contains the field of rational numbers Theorem 1.2 gives the injectivity of σ_{t_0} as a homomorphism from $E_g(K(t))$ to $E_g(t_0)(K)$, so we have to look at the restriction of σ_{t_0} to $E_g^{a,b}(\mathbb{Q}(t))$, which is injective, too.

We first present an example in which the splitting field K of the cubic polynomial $f(x) = x^3 + ax + b$ has class number one with the Galois group the symmetric group \mathbf{S}_3 .

Example 4.3 Let $E = E^{1,1}$ be the elliptic curve over \mathbb{Q} given by equation (4.1) with $a = b = 1$

$$E^{1,1} : y^2 = x^3 + x + 1.$$

Then the elliptic curve E_g has rank two over $\mathbb{Q}(t)$, with free generators the two points $P = P^{1,1}$ and $Q = Q^{1,1}$ listed in Example 4.1. The splitting field K of the polynomial $x^3 + x + 1$ is the sextic field $\mathbb{Q}(q)$ of class number one, generated by the algebraic number q defined as a root of the polynomial $x^6 + 78x^4 + 324x^3 + 1521x^2 + 12636x + 64219$. The two fundamental units of K are

$$\frac{4}{245805}q^5 - \frac{169}{737415}q^4 + \frac{52}{49161}q^3 - \frac{7097}{737415}q^2 - \frac{8728}{49161}q - \frac{13156}{737415},$$

$$\frac{2}{49161}q^5 - \frac{169}{294966}q^4 + \frac{130}{49161}q^3 - \frac{7097}{294966}q^2 + \frac{5521}{98322}q - \frac{6578}{147483}.$$

Further,

$$e_1(t) = - \left(-\frac{2}{35115}q^5 + \frac{169}{210690}q^4 - \frac{26}{7023}q^3 + \frac{7097}{210690}q^2 + \frac{1705}{14046}q + \frac{6578}{105345} \right) \cdot g^{1,1}(t),$$

$$e_2(t) = - \left(\frac{4}{245805}q^5 - \frac{169}{737415}q^4 + \frac{52}{49161}q^3 - \frac{7097}{737415}q^2 - \frac{8728}{49161}q - \frac{13156}{737415} \right) \cdot g^{1,1}(t),$$

$$e_3(t) = - \left(\frac{2}{49161}q^5 - \frac{169}{294966}q^4 + \frac{130}{49161}q^3 - \frac{7097}{294966}q^2 + \frac{5521}{98322}q - \frac{6578}{147483} \right) \cdot g^{1,1}(t).$$

If we choose $t_0 = 3$, it is easy to see that if $h(t)$ is a nonconstant divisor of one of the $\text{rad}_{\mathcal{O}_K[t]}((e_k(t) - e_i(t)) \cdot (e_k(t) - e_j(t)))$ ($i, j, k \in \{1, 2, 3\}$ different) in $\mathcal{O}_K[t]$, then $h(3)$ is not a square in K . Thus Theorem 1.2 is satisfied for $K = \mathbb{Q}(q)$. We conclude that the specialization homomorphism $\sigma_3 : E_g^{1,1}(K(t)) \rightarrow E_g^{1,1}(3)(K)$ is an injection, hence its restriction to $E_g(\mathbb{Q}(t))$ is also an injection. The rank of $E_g^{1,1}(3)(\mathbb{Q})$ is 2. Thus we have shown by using Lemma 4.2(ii) that $E_g = E_g^{1,1}$ has rank two over $\mathbb{Q}(t)$, with free generators P, Q . ■

In the following example the splitting field of $f(x) = x^3 + ax + b$ is a field of class number two, specifically $K = \mathbb{Q}(\sqrt{-5})$. We apply Theorem 1.2 and the result comment from the viewpoint of Theorem 1.3. First we have to choose an adequate $\mathcal{R}_K = \mathcal{R}_{\mathbb{Q}(\sqrt{-5})}$ as in Theorem 1.2. The choice of \mathcal{R}_K in general isn't unique. To reduce the number of divisors in the condition of Theorem 1.2 to a finite number, we choose \mathcal{R}_K with finitely generated group of units.

By the example in [Kn, p.129] we know that we can choose for $\mathcal{R}_{\mathbb{Q}(\sqrt{-5})}$ the principal ideal domain (PID) which is the localization of $\mathcal{O}_K = \mathcal{O}_{\mathbb{Q}(\sqrt{-5})}$ by the multiplicative set $S = \{1, 2, 2^2, 2^3, 2^4, 2^5, \dots\}$, where the group of units is generated by $\mathcal{O}_{\mathbb{Q}(\sqrt{-5})}^\times = \{\pm 1\}$ and 2. So $K = \mathbb{Q}(\sqrt{-5})$, $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$ and $\mathcal{R}_K = S^{-1}\mathcal{O}_K$. For each ideal I of \mathcal{O}_K let us define $I_S := S^{-1}I$. Then I_S is an ideal of \mathcal{R}_K and I_S is proper if and only if $I \cap S = \emptyset$. The non-zero prime ideals of \mathcal{R}_K are exactly $S^{-1}I$ where I goes through non-zero prime ideals of \mathcal{O}_K different from $\mathcal{P} = (2, 1 - \sqrt{-5})$. Since K is the quotient field of the unique factorization domain \mathcal{R}_K , thus we can obtain the irreducible nonconstant factors of a polynomial in $\mathcal{R}_K[t]$ by observing the factorization in $K[t]$.

Example 4.4 Let $E = E^{2,12}$ be the elliptic curve over \mathbb{Q} given by equation (4.1) with $(a, b) = (2, 12)$. Then the elliptic curve $E_g = E_g^{2,12}$ has rank two over $\mathbb{Q}(t)$, with free generators the two points $P = P^{2,12}$ and $Q = Q^{2,12}$ listed in Example 4.1. The splitting field of the polynomial $x^3 + 2x + 12$ is $K = \mathbb{Q}(\sqrt{-5})$ which is of class number two. In this case we have

$$g(t) = g^{2,12}(t) = -2^6 \cdot 3 \cdot (t^2 + 1)(3t^4 + 2t^2 + 2)(3t^4 + 4t^2 + 3)(2t^4 + 2t^2 + 3).$$

Thus we look at the elliptic curve $E_g = E_g^{2,12}$ over $K(t)$. Further from the coefficients we get that the discriminant is equal to $(e_1(t) - e_2(t))^2 \cdot (e_1(t) - e_3(t))^2 \cdot (e_2(t) - e_3(t))^2 =$

$$-2^{40} \cdot 3^6 \cdot 5 \cdot 7^2 \cdot (t^2 + 1)^6 (3t^4 + 2t^2 + 2)^6 (3t^4 + 4t^2 + 3)^6 (2t^4 + 2t^2 + 3)^6,$$

which we have to factor into irreducibles in $\mathcal{R}_K[t]$, only the radical is of importance to get the square-free factors in $\mathcal{R}_K[t]$. One shows $\sqrt{-5}$, $1 \pm \sqrt{-5}$, $3 \pm \sqrt{-5}$ are irreducible elements in the principal ideal domain $\mathcal{R}_K = \mathcal{R}_{\mathbb{Q}(\sqrt{-5})}$ described above. 2 and -1 are invertible elements in \mathcal{R}_K and $\mathcal{R}_K[t]$. We also have $3 = \frac{1}{2}(1 + \sqrt{-5})(1 - \sqrt{-5})$, $2 \pm \sqrt{-5} = -\frac{1}{2}(1 \mp \sqrt{-5})^2$.

We first factor in $K[t]$ to obtain the factorization in $\mathcal{R}_K[t]$. Now it is easy to see that the radical in $\mathcal{R}_K[t]$ of $(e_1(t) - e_2(t)) \cdot (e_1(t) - e_3(t)) \cdot (e_2(t) - e_3(t))$ factors into irreducible elements in the UFD $\mathcal{R}_K[t]$ as $\text{rad}_{\mathcal{R}_K[t]}[(e_1(t) - e_2(t)) \cdot (e_1(t) - e_3(t)) \cdot (e_2(t) - e_3(t))] =$

$$\begin{aligned} &= \sqrt{-5} \cdot (1 + \sqrt{-5}) \cdot (1 - \sqrt{-5}) \cdot (3 + \sqrt{-5}) \cdot (3 - \sqrt{-5}) \cdot (t^2 + 1) \cdot \\ &\cdot \left(t^2 + \frac{1 + \sqrt{-5}}{2}\right) \left(t^2 + \frac{1 - \sqrt{-5}}{2}\right) \left(\frac{1 - \sqrt{-5}}{2}t^2 + 1\right) \left(\frac{1 + \sqrt{-5}}{2}t^2 + 1\right) \cdot \\ &\cdot ((1 + \sqrt{-5})t^2 - (1 - \sqrt{-5})) ((1 - \sqrt{-5})t^2 - (1 + \sqrt{-5})). \end{aligned}$$

So we obtain all nonconstant square-free divisors of $(e_1(t) - e_2(t)) \cdot (e_1(t) - e_3(t)) \cdot (e_2(t) - e_3(t))$ in $\mathcal{R}_K[t]$, we had to take into account -1 and 2 , the two generators of the group of units in \mathcal{R}_K . If we choose $t_0 = 4$ using analogous arguments as in the previous example we conclude $E_g^{2,12}$ over $\mathbb{Q}(t)$ has rank two and free generators P, Q .

Note that $t_0 = 4$ satisfies condition (\mathcal{A}) of Theorem 1.3, which confirms that the specialization at $t_0 = 4$ is injective.

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